# Intersection Theory on Surfaces

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**Abstract**

Intersection theory is a cornerstone of modern algebraic geometry, and the case of surfaces is the simplest and most classical case thereof. This paper develops the intersection pairing and intersection multiplicity, following Hartshorne's "Algebraic Geometry", and presents some examples.

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# 1 Introduction

Intersection theory has interested me since I first learned about algebraic geometry. There's something paradoxical about how simultaneously intuitive and technical it is, the simple ideas it deals with juxtaposed with it's complex and consequential conclusions. It doesn't put tremendous strain on the imagination to see the origins of this discipline dating back to antiquity. After all, intersection theory unites the problem of determining incidence relations with the problem of solving a system of polynomial equations. In our education, we first see this unification accomplished by way the introduction of Cartesian coordinates. However, as these coordinates are generally real, the theory coming from this algebraic geometry would be extremely dissatisfying; there are few, if any, unifying laws to compare the intersections of the vanishing sets of even relatively low-degree polynomials. This is partially mitigated by the introduction of complex coordinates. Over an algebraically closed field, "most" pairs of curves intersect nicely. However, there is still the problem of intersections escaping to infinity; although, for example, a "general" pair of lines in the plane will intersect, there is a single case where they do not. What's more, that case arises as the lines are continuously and very reasonably deformed. To remedy this, we introduce projective geometry. Projective space, the space of all lines through the origin in a vector space, gives a satisfactory answer to the problem of intersections escaping to infinity, but there are still cases where a family of curves, varying "nicely", will exhibit pathological intersection behaviors. For example, as we increase the $c$ coefficient in the parabola $y - x^2 + c$, we have two real roots when $c$ is negative, exactly one when $c$ is zero, and two imaginary roots when $c$ is positive. The answer to this behavior where a reasonably-varying family of curves exhibits non-constant intersection behavior turns out to be counting intersections "with multiplicity", a way of dealing, algebraically, with the "limiting" cases when two or more intersections become close enough together to coincide and occur "on top" of each other. This is the topic of this exposition, and perhaps the man focus of intersection theory in general.

Most of this exposition follows [4], particularly Chapter V, Section 1 with some elements from other chapters. I have endeavored to focus not tireless completeness (recording each proof necessary to accomplish the goals of this paper), but instead to present only the proofs of the results which I found to be unintuitive or in some way central to the understanding of the techniques this paper presents, or for which I found the proofs in [4] exceedingly terse. I have presented, for example, a proof of Bertini's theorem which, although it may not lie within the scope of this paper in the strictest sense, forms the basis for much of the "ethos" of the intersection pairing. Conversely, I have merely cited some technical results used in examples, even though the proofs for those are perhaps far more consequential. In addition to [4], I referenced [6] for algebraic geometry; I worked out of [1] primarily for commutative algebra, supplementing my understanding with [3]. Throughout, I referred to [5].

# 2 Preliminaries

As is standard in algebraic geometry, rings are commutative, unital, and (as I have no interest in being macho; c.f. [6], 10.3.8) noetherian. A prime ideal $\mathfrak{p}$ of a ring $R$ is an ideal such that $R/\mathfrak{p}$ is an integral domain, and a maximal ideal $\mathfrak{m}$ is an ideal such that $R/\mathfrak{m}$ is a field. In particular, the unit ideal $R$ is neither prime nor maximal. Curves and surfaces are varieties of dimension 1 and 2, respectively; varieties are integral and separated schemes of finite type over a ground field $k = \bar{k}$, and I state explicitly the nonsingular hypothesis when I need it.

## 2.1 Some Algebraic Lemmas

**Lemma 2.1.** *Let $R$ be a $k$-algebra and $(f), (g)$ principal ideals for $f, g$ nonzero divisors. Then*

$$\dim_k(R/(fg)) = \dim_k(R/(f)) + \dim_k(R/(g))$$

*supposing all dimensions are finite.*

*Proof.* Consider the sequence of $k$-modules

$$0 \to R/(f) \xrightarrow{m_g} R/(fg) \to R/(g) \to 0$$

Where $m_g$ is multiplication by $g$. By the third isomorphism theorem (c.f. [2], for example), $R/(fg)/(gR/(fg)) \cong R/(g)$ so this sequence is exact. But dimension is additive on short exact sequences, and we are done. $\square$

**Lemma 2.2.** *Let $R$ be a ring, $I$ an ideal of $R$, and $A$ an $R$-algebra. Then*

$$(R/I) \otimes_R A \cong A/(IA)$$

*Proof.* The sequence

$$0 \to I \xrightarrow{i} R \xrightarrow{q} R/I \to 0,$$

where $i$ is the inclusion and $q$ is the quotient map, is exact. Tensoring is exact on the right, so

$$I \otimes_R A \xrightarrow{i \otimes 1} R \otimes_R A \xrightarrow{q \otimes 1} R/I \otimes_R A \to 0$$

is exact. By the isomorphism $r \otimes_R a = 1 \otimes_R ra \to ra$, we obtain that $I \otimes_R A$ is exactly $IA$ and $R \otimes_R A$ is exactly $A$; moreover, we can rewrite $i \otimes 1$ as a map from $IA \to A$ as simply the inclusion (which is injective). Thus

$$0 \to IA \xrightarrow{i\otimes 1} A \xrightarrow{q\otimes 1} R/I \otimes_R A \to 0$$

is exact, and $R/I \otimes A$ is $A/AI$ ☐

**Corollary 2.3.** *Let $R$ be a ring, and $I$ and $J$ be ideals of that ring. Then*

$$(R/I) \otimes_R (R/J) \cong (R/I)/J$$

*Proof.* Note that $R/J$ is an $R$-algebra, and apply Lemma 2.2. ☐

**Lemma 2.4.** *A regular local ring is an integral domain.*

*Proof.* C.F. [1], Lemma 11.23. ☐

We use the following lemma implicitly throughout the rest of this paper.

**Lemma 2.5.** *Let $X$ be a scheme over a field $k$ and $f : Y \to X$ a closed subscheme with sheaf of ideals $\mathscr{I}$. Let $P$ be a geometric point in $X$. Then if $\mathfrak{m}_P$ is the maximal ideal of the local ring $\mathcal{O}_{X,P}$ for $P \in X$, we have that $\mathscr{I}_P \subseteq \mathfrak{m}_P$ if and only if $P \in Y$, i.e. if and only if $f^{-1}(P) \neq \emptyset$.*

*Proof.* Consider the exact sequence of $k$-algebras

$$0 \to \mathscr{I} \xrightarrow{\ker f^\sharp} \mathcal{O}_X \xrightarrow{f^\sharp} i_*\mathcal{O}_Y \to 0.$$

An exact complex of sheaves induces an exact sequence on stalks; in particular, the sequence

$$0 \to \mathscr{I}_P \xrightarrow{\ker f^\sharp} \mathcal{O}_{X,P} \xrightarrow{f^\sharp} (i_*\mathcal{O}_Y)_P \to 0$$

is an exact sequence of $k$-algebras for all $P \in X$. Note that the stalk of the structure sheaf of any scheme cannot be zero at any point in that scheme, as then there would be no neighborhood of that point which was isomorphic to an affine scheme (as the stalks of the structure sheaves of an affine scheme is the localization of a nonzero ring, and thus nonzero). From this we see that $\ker f^\sharp$ cannot surject onto $\mathcal{O}_{X,P}$, and so $\mathscr{I}_P$ must be a proper ideal of $\mathcal{O}_{X,P}$. But as $\mathfrak{m}_P$ is the single maximal ideal, this means $\mathscr{I}_P \subseteq \mathfrak{m}_P$. Conversely, if $f^{-1}(P) = \emptyset$ then $(i_*\mathcal{O}_Y)_P = 0$, because $\Gamma(\emptyset, \mathscr{F}) = 0$ for each sheaf $\mathscr{F}$ by definition, and $\ker f^\sharp$ is an isomorphism. In particular, $\mathscr{I}_P$ is not contained in $\mathfrak{m}_P$. ☐

## 2.2 Bertini's Theorem

Perhaps the foundation of intersection theory on surfaces is characterizing the cases when the set-theoretic intersection is not what we would expect, and demonstrating both that this doesn't happen most of the time (i.e. when the divisors in question are in general position) and that we can always "fix" that failure by some deformation (deforming divisors up to linear equivalence, to be precise). Both these notions are provided by the following theorem.

*Remark* 2.6 (Caution). Here, when $A$ and $B$ schemes embedded into $X$, we write $A \cap B$ for $A \times_X B$. In the future, the intersection $A \cap B$ will denote the set theoretic intersection.

**Theorem 2.7** (Bertini's Theorem)**.** *Let $X$ be a nonsingular closed subvariety of $\mathbb{P}^n_k$, where $k$ is an algebraically closed field. There is a hyperplane $H \subset \mathbb{P}^n_k$, not containing $X$, such that the scheme $H \cap X$ is regular at every point, and the set of all such hyperplanes $H$ is open and dense in $|H|$, when $|H|$ is viewed as a projective space.*

*Proof.* Let $x$ be a closed point in $X$, and let $B_x$ be the set of hyperplanes $H$ for which either $X \subset H$ or $x \in H \cap X$ is a singular point of $H$. Then $H$ is determined by a nonzero global section $f$ of $\mathcal{O}_{\mathbb{P}^n}(1)$, as is every hyperplane. Fix a global section $f_0$ of $\mathcal{O}_{\mathbb{P}^n}(1)$ such that $x \notin H_0 \coloneqq V(f_0)$. We then define a map of $k$-vector spaces

$$\varphi_x : \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to \mathcal{O}_{X,x}/\mathfrak{m}_x^2$$

as follows. Since $f$ is a global section of $\mathcal{O}_{\mathbb{P}^n}(1)$, we have that $f/f_0$ is a regular function on $\mathbb{P}^n - H_0$, which restricts (or pulls back) to a regular function on $X - X \cap H_0$. We let $\varphi_x(f)$ be the image of $f/f_0$ in $\mathcal{O}_{X,x}/\mathfrak{m}_x^2$. The scheme $H \cap X$ is defined in a neighborhood of $x$ by (the restriction of) $f/f_0$, because away from the locus where $f_0$ "vanishes" or "blows up" (where the numerator or denominator lies in the maximal ideal), dividing by it does not change the vanishing locus of $f$. Thus we see that $x \in H \cap X$ if and only if $\varphi_x(f) \in \mathfrak{m}_x$, and $x$ is a nonregular point of $H \cap X$ if and only if the local ring $\mathcal{O}_{H \cap X,x} = \mathcal{O}_{X,x}/\varphi_x(f)$ (this equality follows from an argument in coordinates) is nonregular. This occurs if and only if $\varphi_x(f) \in \mathfrak{m}^2$, because then the quotient $\mathcal{O}_{X,x}/\varphi_x(f)$ will have Krull dimension one less than the Krull dimension of $\mathcal{O}_{X,x}$ (c.f. [1], Corollary 11.18). Meanwhile $\dim_k((\mathfrak{m}_x/\varphi_x(f))/(\mathfrak{m}_x^2/(\varphi_x(f)))) = \dim_k((\mathfrak{m}_x/\mathfrak{m}_x^2)/\overline{(\varphi_x(f))})) = \dim_k \mathfrak{m}_x/\mathfrak{m}_x^2$ as $\varphi_x(f) \in \mathfrak{m}_x^2$, so the ring cannot be regular. This is the only way for the local ring to not be regular, as otherwise $\mathfrak{m}_x$ is generated by $n-1$ generators and Krull's height theorem gives regularity. Thus hyperplanes in $B_x$ are exactly those given by $f \in \ker(\varphi_x)$ (as $f/f_0$ is zero in the local ring of $X$, before passing to the quotient, if and only if $H$ contains $X$). Then since $x$ is a closed point and $k$ is algebraically closed, the maximal ideals of the local rings of projective space over $k$ are generated by linear forms in the coordinates on that projective space. This passes to the local rings of $X$ as well, and so we have that $\varphi_x$ is surjective because each generator of the maximal ideal of the local ring, together with all elements not in the maximal ideal, can be written as $f/f_0$ for some $f$. If $\dim X = r$, then $\dim \mathcal{O}_{X,x}/\mathfrak{m}^2 = \dim \mathfrak{m}/\mathfrak{m}^2 + 1 = r + 1$, as $X$ is nonsingular. Finally, we have $\dim_k \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) = n + 1$, so the dimension of the kernel is $n - r$. Thus the dimension of $B_x$ as a projective space is $n - r - 1$.

Now we consider the complete linear system $|H|$ as a projective space, and study the subset $B$ of $X \times |H|$ given by $B = \{(x, H')|H' \in B_x\} = \bigcup_{x \in X}\{x\} \times B_x$. Then $B$ is the set of all pairs $(x, H')$ where $x$ is a point and $H$ is a hyperplane described by a function $f$ for which $f/f_0$ vanishes to order two at $x$. Vanishing to order two is a closed condition[1], so $B$ is closed; we give $B$ the reduced induced subscheme structure. Further, the map $B \to X$

[1] There is a subtlety here involving choice of $f_0$. Effectively, we can cover $B$ by open sets, to each of which we associate a different $f_0$. We then obtain that $B$ is closed in each open set and thus closed in the ambient space.

given by $(x, H) \mapsto x$ is a map with fiber of dimension $n - r - 1$, and is also surjective. Thus $B$ has dimension $(n - r - 1) + r = n - 1$ (as the dimension of the total space is the dimension of the base plus the dimension of the fiber). As such, the image of the second projection $p_2 : B \to |H|$ can have dimension no more than $n - 1$. Both projections are proper, since $X$ is projective, and thus closed; therefore we see that $\pi_2(B)$ is a closed subset of dimension $\leq n - 1$. Thus $|H| - \pi_2(B)$ is open and dense in $|H|$. $\square$

*Remark* 2.8. The same result holds if $X$ has a finite number of singularities, as the hyperplanes intersecting those singular (closed) points form another proper closed subset of $|H|$, and the intersection of finitely many open dense subsets is again open and dense. Indeed, the hyperplanes intersecting a single point $x$ are exactly those hyperplanes defined by rational functions $f \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ which 'vanish at $x$' (have $\varphi_x(f) \in \mathfrak{m}_x$ for some local trivialization $\varphi_x : \mathcal{O}_{\mathbb{P}^n}(1)|_U \cong \mathcal{O}_{\mathbb{P}^n}|U)$, which is a closed condition and does not describe all hyperplanes in $\mathbb{P}^n$.

# 3 The Intersection Pairing

The goal of this section is to construct the **intersection pairing** mapping $\operatorname{Div} X \times \operatorname{Div} X \to \mathbb{Z}$. As is common in mathematics, especially during the categorical age, we will define our intersection paring not as a construction but as anything that satisfies a list of properties, and then demonstrate that there is a construction which satisfies those properties. The main player here will be divisors. We'll work with Weil divisors, which lend themselves to expression as subschemes of a surface:

*Remark* 3.1. Let $X$ be a surface and $D$ be an effective divisor. We can regard $D$ as a subscheme of $X$ in the following way; let $U_i \cong \operatorname{Spec} A_i$ be an open affine cover over which $D$ is given in $U_i$ as $f_i$. Then let $D'_i$ be the closed subscheme of $U_i$ given by $A_i/(f_i)$. On intersections $U_i \times_X U_j$ we have that $f_i/f_j$ is a regular invertible function, yielding an isomorphism between $D'_i|_{U_j}$ and $D'_j|_{U_i}$. We can glue along these isomorphisms to obtain a closed subscheme $i : D' \to X$. Note further that $\mathscr{L}(-D)$ is given locally on $U_i$ by $f_i\mathcal{O}_{U_i}$; this means that the following short exact sequence is exact for every point $P$

$$0 \to \mathscr{L}(-D)_P \to \mathcal{O}_{U,P} \to \mathcal{O}(D') \to 0.$$

This implies that it is exact globally, and so $\mathscr{L}(-D)$ is the sheaf of ideals of $\mathcal{O}(D')$. We generally identify $D'$ with $D$ in what follows.

**Definition 3.2.** Let $C$ and $D$ be curves on a surface $X$. Then $C$ and $D$ **intersect transversely at a point** $P$ if $P$ lies in $C \cap D$ and for locally defining functions $f, g$ for $C, D$ respectively in a neighborhood of $P$ restrict to functions $f|_P$ and $g|_P$ which generate the maximal ideal $\mathfrak{m}_P$. The divisors $C$ and $D$ **intersect transversely** if they intersect transversely at every point in $C \cap D$.

*Remark* 3.3. The main requirement that we expect an intersection pairing to have is that it should generalize the notion of counting intersection points with a transversal hypothesis; transversality captures the notion of "passing through without multiplicity", or the "ideal" intersection. For this reason, we require our intersection pairing to satisfy

**Axiom 1.** If $C$ and $D$ are nonsingular, irreducible, effective divisors in a surface $X$ which intersect transversely, then:

$$C \cdot D = \#(C \cap D)$$

*Remark* 3.4. This Axiom 1 differs slightly from the corresponding axiom in [4], as it requires only irreducible effective divisors satisfy $C \cdot D = \#(C \cap D)$. This is justified because we still obtain that the intersection pairing is unique, so any intersection pairing satisfying [4]'s stronger condition must also satisfy our weaker condition, and thus be the same as ours. The reason we make this change is to allow an easier proof of Lemma 3.13.

*Remark* 3.5. The reason we want to count exactly the set-theoretic intersection points *only* when we have transversality is that we want our intersection pairing to be invariant under some sort of deformation; in particular, linear equivalence of varieties. For this reason, we also require our pairing descend to the Picard group; that is, to only depend on linear equivalence class. Finally, we also require our intersection pairing to be symmetric and $\mathbb{Z}$-bilinear, because (intuitively) the "number of points" in the intersection of two varieties shouldn't depend on the order you intersect them, and should be additive in each of the curves.

**Definition 3.6.** An **intersection pairing** is any symmetric, bilinear pairing $\cdot : \operatorname{Div} X \times \operatorname{Div} X \to \mathbb{Z}$ which satisfies Axiom 1 and induces a well-defined pairing on $\operatorname{Pic} X$.

We will soon show that these properties determine *the* intersection pairing entirely. Before this, however, we need some preliminary results. We begin with some lemmas which we will use to work with the fiber product as the scheme-theoretic intersection.

**Lemma 3.7.** *If $A, B$ are closed subschemes and $C$ an open subscheme of $X$, then*

1. $A|_C \cong A \times_X C$
2. $(A \times_X C) \times_X (B \times_X C) \cong A \times_C B$.

*Proof.* Follows from `https://stacks.math.columbia.edu/tag/01JO` and Corollary 2.3 (for (2)). ☐

*Remark* 3.8. Lemma 3.7, Item 1 could (and probably should) be definitional, but we are attempting to follow conventions from [4].

**Lemma 3.9.** *Let $C$ and $D$ be divisors which share no irreducible component in a smooth surface $X$, and suppose $C \times_X D$ contains a point $P$ and is regular at that point. Then $C$ meets $D$ transversely at that point.*

*Proof.* Suppose $C$ and $D$ are defined respectively by $f$ and $g$ in an affine neighborhood $U \cong \operatorname{Spec} A$ of the point $P$. Suppose $\mathcal{O}_{C \times_X D, P}$ is regular. Then since $C$ and $D$ share no irreducible component, $C \times_X D$ has dimension 0 and so $\mathfrak{m}_{C \times_X D, P}/\mathfrak{m}^2_{C \times_X D, P}$ has dimension zero as a vector space over $k$, meaing $\mathfrak{m}_{C \times_X D, P} = \mathfrak{m}^2_{C \times_X D} = (0)$ and thus $\mathcal{O}_{C \times_X D, P} \cong k$. Recall, however, that $\mathcal{O}_{C \times_X D, P} \cong (A/(f) \otimes_A A/(g))_P \cong (A/(f,g))_P$ by Corollary 2.3. Moreover, by [1], Corollary 3.4 (localization commutes with quotients[2]) $(A/(f,g))_P \cong A_P/(f,g)$. Since this is a field, $(f,g)$ is a maximal ideal; since $A_P$ is a local ring, that maximal ideal is unique and must be $\mathfrak{m}_P$. ☐

[2]Although [1] shows this only for modules, one can generalize the argument to algebras.

We will need the irreducible hypothesis to prove the lemma which will allow us to move varieties and compute the intersection pairing; however, we will not always necessarily have it. We will have access, by way of Theorem 2.7, to a large supply of smooth varieties, and so we show that this suffices.

**Lemma 3.10.** *A smooth, connected scheme is irreducible.*

*Proof.* Suppose $X$ is connected and smooth but not irreducible. Then two of the irreducible components $Y$ and $Y'$ must intersect at some point $P$. Each neighborhood of $P$ then intersects both components, and thus there is an open affine $U \cong \operatorname{Spec} A$ which intersects both components. Then there exist nontrivial ideals $I$ and $J$ such that $Y \cap U \cong V(I)$ and $Y' \cap U \cong V(J)$, with $V(I) \cup V(J) \cong U$, or equivalently $V(IJ) \cong U$. But this means $IJ$ is contained in every nontrivial ideal, and thus contained in the nilradical. We can perform this for every affine open set containing $P$, and each open set containing $P$ contains an affine open containing $P$, so the local ring $\mathcal{O}_{X,P}$ at this point contains a zero divisor pair $de = 0$. By Lemma 2.4, then, we obtain a contradiction as the local ring must be regular, and the lemma is proven. $\square$

We now prove the main lemma allowing us to move vareties in our surface in order to compute the intersection pairing. This can be viewed, perhaps, as the lemma which deals with the more pathological cases, in contrast to a following lemma.

**Lemma 3.11.** *Let $C_1, ..., C_r$ be irreducible curves on the surface $X$, and let $D$ be a very ample divisor. Then there is an irreducible nonsingular divisor $D'$ which is linearly equivalent to $D$, and meets each of the $C_i$ transversely.*

*Proof.* Let $\iota : X \to \mathbb{P}^n$ be the embedding with respect to which $D$ is very ample. Then each divisor linearly equivalent to $D$ has an associated line bundle which is isomorphic to the pullback of the hyperplane bundle on $\mathbb{P}^n$ through $\iota$. By Theorem 2.7 the set of hyperplanes $H$ such that $H \times_{\mathbb{P}^n} X$ is nonsingular and $H \times_{\mathbb{P}^n} C_i$ is also nonsingular for all $C_i$ is a finite intersection of dense sets, and thus dense. Furthermore, $H \times_{\mathbb{P}^n} X$ has associated line bundle $\iota^*\mathcal{O}(H)$ which is isomorphic $\mathscr{L}(D)$; thus $H \times_{\mathbb{P}^n} X$ is linearly equivalent to $D$. Since $H \times_{\mathbb{P}^n} X$ is connected and smooth, by Lemma 3.10 we obtain irreducibility. Finally, since $X$ contains $C_i$, we have that $H \times_{\mathbb{P}^n} C_i$ embedds into $H \times_{\mathbb{P}^n} X$, and in particular, $H \times_{\mathbb{P}^n} X \times_{\mathbb{P}^n} C_i = H \times_{\mathbb{P}^n} C_i$. As such, the fact that the intersections $H \times_{\mathbb{P}^n} X \times_{\mathbb{P}^n} C_i$ contain only regular points means by Lemma 3.9 the intersections are transversal and we obtain the required result. $\square$

**Lemma 3.12.** *Let $D$ be an effective divisor on a smooth curve $C$, viewed as a subscheme by Remark 3.1. If $D$ is given near a point $P \in C$ by $f$, we have that $\dim_k \mathcal{O}_D = \eta_P(f)$. Thus*

$$\deg D = \sum_{P \in \operatorname{Supp} D} \dim_k \mathcal{O}_D$$

*Proof.* Since $D$ is effective, in each local ring we have $f = g_P^n$ where $g_P$ is the generator of the maximal ideal and $n \geq 0$. Thus we have the exact sequence:

$$0 \to (g_P^n)\mathcal{O}_{C,P} \to \mathcal{O}_{C,P} \to \mathcal{O}_D \to 0$$

and since $\mathcal{O}_{C,P}$ is a principal ideal domain, $\mathcal{O}_D$ has dimension $n$. The second statement follows from this and the fact that

$$\deg D = \sum_{P \in \operatorname{Supp} D} \eta_P(f_P)$$

where $f_P$ is a local defining function near $P$. $\square$

We now prove the main lemma allowing us to relate the set-theoretic intersection to the intersection pairing or scheme theoretic intersection. This lemma could be viewed as the input insuring that our intersection pairing matches up well with "reality" - i.e, in the transverse case, it matches the set theoretic intersection.

**Lemma 3.13.** *Let $C$ be a curve on a surface $X$, and let $D$ be any curve meeting $C$ transversely. Then*

$$\#(C \cap D) = \deg_C(i^*(\mathscr{L}(D)))$$

*where $i$ is the inclusion of $C$ into $X$.*

*Proof.* Note first that the underlying sets of $C \times_X D$ is $C \cap D$, by the universal property of the fiber product. Moreover, since $D$ and $C$ meet transversely, they cannot share an irreducible component; if they did, at any point along that component the ideal generated by one locally defining function would be contained in the other (and thus not generate the maximal ideal at that point, as otherwise the Krull dimension of the local ring would be at most 1, the dimension of a principal ideal). Thus $C \cap D$ is a discrete set of points. Furthermore, we may suppose that $C$ and $D$ are both smooth in a neighborhood of each of the intersections, as the maximal ideal is given by two generators and thus $\mathfrak{m}/\mathfrak{m}^2$ is two dimensional. This allows us to apply Lemma 3.12. Since $\mathscr{L}(-D)$ is the sheaf of ideals for $D$, we have the short exact sequence

$$0 \to \mathscr{L}(-D) \to \mathcal{O}_X \to i_*\mathcal{O}_D \to 0$$

where $j : D \to X$ is the inclusion. Since closed embeddings are flat, the pullback through $i$ is exact, so we obtain

$$0 \to i^*\mathscr{L}(-D) \to \mathcal{O}_C \to i^*j_*\mathcal{O}_D \to 0.$$

Let $U_i \cong \operatorname{Spec} A_i$ be an affine open cover of $X$, such that $C$ is given on $U_i$ by $A_i/(f_i)$ and $D$ by $A_i/(g_i)$ (Remark 3.1). Then $\mathcal{O}_D$ is given on $U_i$ by $A_i/(g_i)$, and (since $i$ is patched together from the quotient maps $i^\sharp : A \to A/(f_i)$), we have that on stalks

$$(i^*j_*\mathcal{O}_D)_P \cong i^{-1}(j_*\mathcal{O}_D) \otimes_{\mathcal{O}_X} \mathcal{O}_C \cong (A_i/(g_i))_P \otimes_A (A/(f_i))_P,$$

(where we view $P$ as a prime ideal of $A$), and finally the right hand side is isomorphic to $(A_i)_P/(g_i, f_i)$, by Corollary 2.3 and the fact that localization commutes with both the tensor product and quotients. But $(A_i)_P/(g_i, f_i) \cong k$, by the transversality hypothesis. Applying Lemma 3.12 yields that $i^*\mathscr{L}(-D)$ is the sheaf of ideals for a divisor which has degree

$$\sum_{P \in C \cap D} \dim_k(k) = \#(C \cap D).$$

Since pullbacks commute with duals for invertible sheaves, $i^*\mathscr{L}(-D)$ is also the sheaf of ideals for the divisor associated to $i^*\mathscr{L}(D)$, and we are done. $\square$

*Remark* 3.14. At this point, the only difficult task is demonstrating that if $C \sim C'$ both meet $D$ transversely, then $\#(C \cap D) = \#(C' \cap D)$. If this holds, then Lemma 3.11 gives us the ability to perturb very ample divisors by linear equivalence in order to obtain a transversely intersecting divisor, which in turn yields a well-defined formula. Then it just remains to extend the definition to non-ample divisors, which we do by finding ample divisors linearly equivalent to the non-ample divisors and intersecting those, using the properties of the intersection product on ample divisors to compute a meaningful result. In particular, we are ready to prove the

**Theorem 3.15.** *There exists a unique intersection pairing $\cdot$ on any smooth projective surface $X$.*

*Proof.* We begin with uniqueness. Suppose $H$ is an ample divisor on $X$. Given any pair of divisors $C, D$ on $X$, we can find (by the definition of an ample divisor) an integer $k > 0$ such that $\mathscr{L}(C + kH), \mathscr{L}(D + kH)$, and $\mathscr{L}(kH)$ are all generated by global sections. We can also find an $l$ such that $lH$ is very ample, meaning that $n = k + l$ has the property $C + nH, D + nH$, and $nH$ are all very ample. Then by Lemma 3.11 we can choose nonsingular, irreducible curves $C', D', E', F'$ which satisfy

$$C' \sim C + nH \tag{1}$$
$$D' \sim D + nH, \text{ transversal to } C' \tag{2}$$
$$E' \sim nH, \text{ transversal to } D' \tag{3}$$
$$F' \sim nH, \text{ transversal to } C' \text{ and } E' \tag{4}$$

Then in particular we have $C' - E' \sim C$ and $D' - F' \sim D$. Then we have, by bilinearity on the Picard group that any intersection product, if it exists, must satisfy

$$C \cdot D = (C' - E') \cdot (D' - F') = C' \cdot D' - C' \cdot F' - E' \cdot D' + E' \cdot F'$$

But then by Axiom 1, since $D'$ and $F'$ are transversal to $C'$, $F'$ is transversal to $E'$, and $E'$ is transversal to $D'$, we have

$$= \#(C' \cap D') - \#(C' \cap F') - \#(E' \cap D') + \#(E' \cap F')$$

Which is a well-defined unique integer independent of intersection pairing. Thus any intersection pairing is unique.

We now seek to show that such a pairing exists. First we work only on the set of very ample divisors. We wish to define $C \cdot D$ to be $\#(C' \cap D')$, where $C' \sim C$ is nonsingular and irreducible and $D' \sim D$ is nonsingular, irreducible, and transverse to $C'$ (both by Lemma 3.11). However, this is only well-defined if the condition in Remark 3.14 holds, namely if $D'' \sim D'$ is nonsingular and irreducible then $\#(C' \cap D'') = \#(C' \cap D')$. Recall (Lemma 3.13) that $\#(C' \cap D') = \deg_{C'}(\mathscr{L}(D') \otimes \mathcal{O}_{C'})$; but this is the same as $\deg_{C'}(\mathscr{L}(D'') \otimes \mathcal{O}_{C'})$, and this is then $\#(C' \cap D'')$. This yields that the definition is well defined, and moreover gives us Axiom 1 and that our pairing depends only on linear equivalence class. Furthermore, as degree is additive and the number of points in an intersection is symmetric, this definition gives a symmetric bilinear pairing. Thus we have an intersection pairing $C \cdot D = \#(C' \cap D')$ for $C, D$ very ample divisors. Now suppose

$C$ and $D$ are not ample. We then repeat the process outlined in the proof of uniqueness, using Lemma 3.11 to write $C = C' - E'$ and $D = D' - F'$ for $C'$, $D'$, $E'$ and $F'$ which satisfy conditions (1) – (4). We define

$$C \cdot D = C' \cdot D' - C' \cdot F' - E' \cdot D' + E' \cdot F'$$

Any other choice of $C', E', D'$ and $F'$ satisfying the requisite relations must be the same up to linear equivalence, and thus give the same result. Similarly, descent to the Picard group follows from the corresponding property on ample divisors, as does symmetric bilinearity. Finally, to verify Axiom 1, we suppose $C$ and $D$ are nonsingular, irreducible curves which may not be ample divisors. If

$$C \cdot D = C' \cdot D' - C' \cdot F' - E' \cdot D' + E' \cdot F',$$

for divisors satisfying (1) – (4) above, then

$$C \cdot D = \#(C' \cap D') - \#(C' \cap F') - \#(E' \cap D') + \#(E' \cap F'),$$

which means by Lemma 3.13 we have

$$C \cdot D = \deg_{C'}(i^*\mathscr{L}(D')) - \deg_{C'}(i^*\mathscr{L}(F')) - \deg_{E'}(i^*\mathscr{L}(D')) + \deg_{E'}(i^*\mathscr{L}(F')).$$

By additivity, definition, and Lemma 3.13:

$$C \cdot D = \deg_{C'}(i^*\mathscr{L}(D)) - \deg_{E'}(i^*\mathscr{L}(D)) = \#(C' \cap D) - \#(E' \cap D)$$

$$= \deg_D(j^*\mathscr{L}(C')) - \deg_D(j^*\mathscr{L}(E')) = \deg_D(j^*\mathscr{L}(D' - E')) = \deg_D(j^*\mathscr{L}(C)) = \#(C \cap D)$$

And we are done. □

**Corollary 3.16** (Bézout's Theorem)**.** *Two curves $C$ and $D$ in $\mathbb{P}^2$ which share no irreducible component intersect with multiplicity* $(\deg C)(\deg D)$ *times.*

*Proof.* Recall that the Picard group of $\mathbb{P}^2$ is $\mathbb{Z}$, and is generated by any line $l$. Thus if we let $\deg C = c$ and $\deg D = d$, we may say (since degree is an isomorphism from the Picard group to $\mathbb{Z}$) that $C \sim cl$ and $D \sim dl'$ for some other line $l' \neq l$, which we may suppose intersects $l$ transversely. Then $l \cdot l' = \#(l \cap l') = 1$, since by definition any pair of lines meet once in the projective plane. Then $C \cdot D = cl \cdot dl' = cd(l \cdot l') = cd$, and we are done. □

# 4 Intersection Multiplicity

The intersection pairing is interesting in and of itself, but it is difficult to compute with; moving divisors explicitly is not always easy. The intersection pairing also lacks some geometric motivation. In this section we attempt to remedy both these issues with the following

**Definition 4.1.** The **intersection multiplicity** of two effective divisors $C$ and $D$ with no common irreducible component at a point $P \in C \cap D$ is $\dim_k \mathcal{O}_{X,P}/(f, g)$, where $f$ and $g$ are local defining functions for $C$ and $D$ respectively in a neighborhood of $P$. We write the intersection multiplicity of $C$ and $D$ at a point $P$ as $(C \cdot D)_P$.

Before we utilize this construction and demonstrate its relationship to the intersection pairing, we require prove one elementary result regarding sheaf cohomology;

**Lemma 4.2.** *Let $\mathscr{F}$ be a sheaf on a topological space $X$ supported in a finite set of discrete points $\{x_i\}$. Then:*

$$H^j(X, \mathscr{F}) = \begin{cases} \bigoplus_i \mathscr{F}_{x_i} & j = 0 \\ 0 & j > 0 \end{cases}$$

*Proof.* Let $\iota_i$ denote the inclusion of the point $x_i$ into $X$. Then $\mathscr{F} \cong \oplus_i (\iota_i)_* \mathscr{F}_{x_i}$, with the isomorphism given by taking any section $s \in \Gamma(U, \mathscr{F})$ to $s|_{x_i}$ if $x_i \in U$ and 0 otherwise. This induces an isomorphism on stalks, because each $x_i$ has a neighborhood containing none of the other $x_i$, and so on each stalk the map is the identity.

Now we have that $H^j(X, \mathscr{F}) \cong \oplus_i H^j(X, (\iota_i)_* \mathscr{F}_{x_i})$, since cohomology commutes with direct sums. But we have that $(\iota_i)_*$ is exact, so the Leray spectral sequence

$$H^p(X, R^q(\iota_i)_* \mathscr{F}_{x_i}) \implies H^{p+q}(\{x_i\}, \mathscr{F}_{x_i})$$

degenerates on page two (in the sense that the row $q = 0$ is the only nonzero row), yielding isomorphisms

$$H^p(X, (\iota_i)_* \mathscr{F}_{x_i}) \cong H^p(\{x_i, \mathscr{F}_{x_i})$$

the global sections functor is exact on a point, because the only open cover of the point is the point itself, so we are done. □

**Proposition 4.3.** *Suppose $C$ and $D$ are effective divisors on $X$ with no irreducible component in common. Then*

$$C \cdot D = \sum_{P \in C \cap D} (C \cdot D)_P$$

*In particular, since the left hand quantity is finite, the right hand quantity is as well.*

*Proof.* We will show that

$$\sum_{P \in C \cap D} (C \cdot D)_P$$

satisfies the axioms of the intersection pairing, and is thus (by uniqueness) equal thereto. First, the pairing is clearly symmetric, so symmetry is satisfied. If $C$ and $D$ are nonsingular, irreducible, and intersect transversely at each point $P$ where they intersect, then $(C \cdot D)_P = \dim_k(\mathcal{O}_{X,P}/(f, g))$. But $(f, g) = \mathfrak{m}_P$, so $\mathcal{O}_{X,P}/(f, g) = k$ and $(C \cdot D)_P = \dim_k(\mathcal{O}_{X,P}/(f, g)) = 1$. Thus for curves $C$ and $D$ which are irreducible and intersect transversely, we have

$$\sum_{P \in C \cap D} (C \cdot D)_P = \sum_{P \in C \cap D} 1 = \#(C \cap D)$$

and Axiom 1 is satisfied as well. If $C$ and $C'$ have no irreducible component in common with $D$, then neither does $C + C'$; furthermore, if $C$ is locally defined near a point $P$ by

$f_P$ and $C'$ by $f'_P$, we have that $C + C'$ is locally defined by $f_P f'_P$. Suppose also that $D'$ is defined near a point $P$ by $g_P$. In this case, we have

$$\sum_{P \in C \cap D} ((C + C') \cdot D)_P = \sum_{P \in C \cap D} \dim_k(\mathcal{O}_{X,P}/(f_P f'_P, g_P))$$

then by Lemma 2.1 we have that

$$\dim_k(\mathcal{O}_{X,P}/(f_P f'_P, g_P)) = \dim_k(\mathcal{O}_{X,P}/(f'_P, g_P)) + \dim_k(\mathcal{O}_{X,P}/(f_P, g_P)),$$

so summing over each point in the intersection yields that

$$\sum_{P \in C \cap D} \dim_k(\mathcal{O}_{X,P}/(f_P f'_P, g_P)) = \sum_{P \in C \cap D} \dim_k(\mathcal{O}_{X,P}/(f'_P, g_P)) + \dim_k(\mathcal{O}_{X,P}/(f_P, g_P)).$$

Then by definition we may conclude that

$$\sum_{P \in C \cap D} ((C + C') \cdot D)_P = \sum_{P \in C \cap D} (C \cdot D)_P + (C' \cdot D)_P$$

and bilinearity is satisfied. Thus it suffices to show that this expression is invariant under linear equivalence; that is, it induces a well-defined map on the Picard group. Consider $C \cap D$ as a subscheme of $C$, and note that its structure sheaf will be $\mathcal{O}_C/\mathscr{L}(-D) \otimes \mathcal{O}_C$, because $\mathscr{L}(-D)$ is the ideal sheaf of $D$. This is a sheaf with stalk equal to zero over any closed point which is not in $D$, and equal to $\mathcal{O}_{X,P}/(f, g)$ over any point in the intersection. This gives us the following exact sequence of sheaves over $C$,

$$0 \to \mathscr{L}(-D) \otimes \mathcal{O}_C \to \mathcal{O}_C \to \mathcal{O}_{D \cap C} \to 0,$$

and also by Lemma 4.2 that

$$\dim H^0(X, \mathcal{O}_{C \cap D}) = \sum_{P \in C \cap D} (C \cdot D)_P.$$

But simultaneously, we have that (as the Euler characteristic is additive on exact sequences)

$$-\chi(\mathscr{L}(-D) \otimes \mathcal{O}_C) + \chi(\mathcal{O}_C) = \chi(\mathcal{O}_{D \cap C}),$$

where $\chi(\mathscr{F}) = \sum_i (-1)^i \dim_k(H^i(X, \mathscr{F}))$ as usual. But higher cohomology of sheaves with discrete support is trivial (Lemma 4.2), so

$$-\chi(\mathscr{L}(-D) \otimes \mathcal{O}_C) + \chi(\mathcal{O}_C) = \dim_k H^0(X, \mathcal{O}_{C \cap D}) = \sum_{P \in C \cap D} (C \cdot D)_P,$$

since the right hand side thus depends only on the sheaf $\mathscr{L}(D)$ up to isomorphism, or (equivalently) the divisor $D$ up to linear equivalence, the left hand side does as well. This combined with symmetry shows that this pairing induces a well defined pairing on the Picard group, and we are done. □

This construction is especially conducive to working out examples:

*Example* 4.4 (Quadratic). Let $X = \mathbb{P}^2$, $C$ be the vanishing set of the global section $x^2 - yz$ of $\mathcal{O}(2)$, and $D$ be the vanishing set of the global section $y$ of $\mathcal{O}(1)$ (here by vanishing of $f$ we mean the set of all points $P$ where $f \in \mathfrak{m}_P$). The intersection $C \cap D$ contains all the points where $y$ and $x^2 - yz$ are in $\mathfrak{m}_P$. Consider the affine patch $\{z \neq 0\}$, and suppose that $C \cap D$ has a point $P$ outside that open set. Clearly the point $P = [0 : 0 : 1]$ in this patch is in $C \cap D$. We seek to compute the intersection multiplicity at this point. For this it will suffice to work in the affine plane, with the equations $x^2 - y = 0$ and $y = 0$. Then we compute $\mathcal{O}_{X,P} = k[x, y]_{(x,y)}$ and

$$\mathcal{O}_{X,P}/(x^2 - y, y) = k[x]_x/(x^2) \cong k \oplus k,$$

and so $(C \cdot D)_P = 2$. This confirms our intuition from calculus that the multiplicity of the quadratic's zero is 2. But are there no other points in the intersection? There are no other intersections in this affine patch, so it remains only to check the points at infinity. If $z \in \mathfrak{m}_P$, then $x^2 - yz \in \mathfrak{m}_P$ implies $x \in \mathfrak{m}_P$ by primality, so if $y \in \mathfrak{m}_P$ we have that $x, y, z \in \mathfrak{m}_P$. But $(x, y, z) \notin \operatorname{Proj} k[x, y, z]$ as we exclude the irrelevant ideal, so there is no intersection at infinity and we have verified Bézout's theorem in this case.

*Example* 4.5. Blow up a point $O$ in $\mathbb{P}^2$ to obtain a birational map $\pi : \widetilde{\mathbb{P}}^2 \to \mathbb{P}^2$ and let $E$ denote the exceptional divisor. Let $L$ be a line in $\widetilde{\mathbb{P}}^2$; that is, an irreducible subscheme of $\widetilde{\mathbb{P}}^2$ which is isomorphic to $\mathbb{P}^1$. Suppose without loss of generality (up to an automorphism) that $O$ is the origin in our coordinates. Then if $L$ lies at the line at infinity, the intersection with $E$ is trivial; we see this because $\pi$ is an isomorphism on $L$ and $\pi(L) \not\ni O$, thus $L \cap E = \emptyset$. Otherwise, we can work in an affine patch centered at $O$, and write $L$ as one of the (possibly multiple) irreducible components of $V(ax + by + c, xw - yz)$ for $a, b$ constant. When $c \neq 0$, we have that $V(ax + by + c, xw - yz)$ is irreducible and does not meet $E$ ($E$ is the locus $x = y = 0$; setting $x = y = 0$ and $ax + by + c = 0$ simultaneously implies that $c = 0$). This set forms an open and dense subset of the set of lines in $\widetilde{\mathbb{P}}^2$, so we may say the generic line does not intersect $E$. But what if $c = 0$? Then we can assume without loss of generality that $a \neq 0$ and write $V(x - by, xw - yz) = V(x - by, y(bw - z))$. This thus includes two irreducible components, $L \coloneqq V(x - by, bw - z)$ and $V(x - by, y) = E$. Passing to the affine patch $w \neq 0$, we obtain that $L = V(x - by, z - b)$. As such, $L \cap E$ is the point $x = y = 0, z = b$. At this point, we compute

$$\dim_k \mathcal{O}_{\widetilde{\mathbb{P}}^2,P}/(f, g) = \dim_k(k[x, y, z]/(b - z))/(x - by, y) = 1$$

Where we quotient first by $(b - z)$ because the surface we're working in is given in the local coordinates by $b - z$, and then we quotient by the defining functions for the two ideals. Thus the line intersects the exceptional divisor transversely at exactly one point; we then have

$$L \cdot E = \begin{cases} 1 & \pi(L) \text{ meets } O \\ 0 & \text{otherwise} \end{cases}$$

*Remark* 4.6. In the previous two examples, we talk rather loosely about coordinates equaling zero, using more of the language of varieties than of schemes. Effectively, when we say "a point where $x = 0$" we mean "a point $P$ such that $x$ lands in the maximal ideal at $P$ after the action of the appropriate line bundle's local isomorphism to the structure sheaf". We can then speak of substitution and generally use the standard rules of algebra, by working in affine coordinates and using the substitution laws of a quotient of rings.

## 5 Self Intersections and Blowups

Intersection theory has myriad applications across a wide range of disciplines in algebraic geometry and beyond. For the this paper, we wish to introduce only one of these. One of the interesting features of the intersection pairing is the ability to compute "how many times a curve intersects itself":

**Definition 5.1.** Let $C$ be a divisor in a surface $X$. The **self intersection** of $C$ in $X$ is $C^2 = C \cdot C$.

Of course, this construction measures the number of times a curve intersects a slightly deformed copy of it, and finds an interesting application in birational geometry. It is well-known that every birational transformation of surfaces factors through a finite number of blowups of points. As such, characterizing birational transformations is a fruitful pursuit. It turns out that self intersection is the correct way to go about this; the exceptional divisor of every blowup of a point has self intersection $(-1)$, and conversely, every curve with self intersection $(-1)$ is the exceptional divisor of such a blowup. Here we only scratch the surface of this interesting theory with a very simple example, however, the techniques we use should generalize well.

**Lemma 5.2.** *Let $\pi : \tilde{X} \to X$ be any birational transformation of surfaces which is an isomorphism away from a point $P$ on $X$, such that $\pi^{-1}(P) = E$ is a divisor (in particular, the blowup of a point). Let $C$ and $D$ be linearly equivalent curves on $X$. Then $\pi^{-1}(C)$ and $\pi^{-1}(D)$ are linearly equivalent.*

*Proof.* Consider $\pi^{-1}(C)$ and $\pi^{-1}(D)$ as sets. These will be closed, as $\pi$ is a continuous map; give them the reduced induced subscheme structure. On a sufficiently fine open cover $\{U_i\}$, we have locally defining functions $f_i$ for $C$ and $g_i$ for $D$. Pulling those functions back by $\pi^\sharp$, we have that (since $\pi^\sharp$ is a homomorphism of local rings) $\pi^\sharp(f_i) \in \mathfrak{m}_{P,X}$ if and only if $f_i \in \mathfrak{m}_P$ and symmetrically for $g_i$. Thus the points cut out by $\pi^\sharp(f_i)$ are exactly $\pi^{-1}(C)$, and similarly for $D$. Furthermore, since we give the preimage the reduced subscheme structure, it is reduced; but so are $C$ and $D$. In particular, no irreducible subscheme appears in either with multiplicity greater than one. From this we conclude that $f^{-1}(C)$ is defined by $f^\sharp(f_i)$ locally, and similarly for $f^{-1}(D)$ and $f^\sharp(g_i)$. But $g_i = hf_i$ by the linear equivalence of $C$ and $D$ for a rational function $h$ and all $i$, so we have $\pi^\sharp(g_i) = \pi^\sharp(hf_i) = \pi^\sharp(h)\pi^\sharp(f_i)$. $\pi^\sharp(h)$ is a rational function, and so we have linear equivalence. $\square$

*Example* 5.3. Let $E$ be the exceptional divisor of $\mathbb{P}^2$ blown up at the origin $(0,0) = V(x,y)$[3]. Let $C = V(x) \subset \mathbb{P}^2$ and $D = V(y) \subset \mathbb{P}^2$. Then $C \cdot D = 1$ as two lines meet in $\mathbb{P}^2$ at exactly one point. Further consider $\tilde{C}$ and $\tilde{D}$, given by the strict transform of $C$ and $D$. The idea of this example is to argue that since $\pi$ is an isomorphism away from $E$, the intersection pairing of two divisors linearly equivalent $C$ and $D$ which do not meet $P$ is the same as the intersection pairing of the pre-image of those divisors under $\pi$, and then to use Lemma 5.2 to compare those to the pre-images of $C$ and $D$ under $\pi$. Since the pre-image of $C$ and $D$ include $E$, we'll then compute $\pi^{-1}(C) \cdot \pi^{-1}(D) = (\tilde{C} + E) \cdot (\tilde{D} + E)$ to deduce the intersection pairing of $E$ with itself.

[3]Here we choose, arbitrarily, that the origin is the origin of the $z \neq 0$ affine patch.

Let $A$ denote the affine patch of $\mathbb{P}^2$ containing the origin (clearly the divisors cannot intersect away from the origin because the projection $\pi$ is an isomorphism away from the origin). Then the strict transform $\tilde{C}$ of $C$ can be described as a subset of $\mathbb{A}^2 \times \mathbb{P}^1$ (with affine coordinates $(x, y)$ and projective coordinates $[z : w]$) as $\tilde{C} = V(x, xz - yw)$. Similarly, $\tilde{D}$ can be described as $\tilde{D} = V(y, xz - yw)$. In the affine patch $z \neq 0$ we may dehomogenize to obtain $\tilde{C} = V(x, x - yw) = V(yw, x - yw)$. This decomposes into two irreducible components; $V(w, x - yw)$ is the image of the strict transform $\tilde{C}$ under this dehomogenization and $V(y, x - yw)$ is the exceptional divisor. They intersect (transversely) at $V(x, y, w)$. When $w \neq 0$ there is no intersection; we dehomogenize to obtain $V(x, xz - y)$, which yields the single irreducible component $V(x, y)$ the exceptional divisor. A symmetric argument shows that $\tilde{D}$ intersects the exceptional divisor (transversely) at the point $V(x, y, z)$, and so the curves do not intersect. (They cannot intersect away from the exceptional divisor, and they do not intersect on the exceptional divisor). They are also effective divisors, because they are given by curves. Thus we obtain that $\tilde{C} \cdot \tilde{D} = 0$ by Proposition 4.3.

Consider $\tilde{C} + E$ and $F = \pi^{-1}(V(x - 1))$. By Lemma 5.2, $\pi^{-1}(C) = E + \tilde{C}$ is linearly equivalent to the line $\pi^{-1}(V(x - 1))$, and $\pi^{-1}(D) = E + \tilde{D}$ is linearly equivalent to $\pi^{-1}(V(y - 1))$. Since $\pi$ is an isomorphism away from $E$ and neither $\pi^{-1}(V(x - 1))$ nor $\pi^{-1}(V(y - 1))$ meet $E$, we have that their intersection pairing is just the intersection pairing $V(x - 1) \cdot V(y - 1)$ in $\mathbb{P}^2$, which is one. This means that

$$(\tilde{C} + E) \cdot (\tilde{D} + E) = 1.$$

But we already have that $\tilde{C} \cdot E = \tilde{D} \cdot E = 1$, and that $\tilde{C} \cdot \tilde{D} = 0$, so we obtain

$$\tilde{C} \cdot \tilde{D} + \tilde{C} \cdot E + \tilde{D} \cdot E + E \cdot E = 1,$$

or $0 + 1 + 1 + E \cdot E = 1$. Thus $E \cdot E = -1$.

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